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Switching of Asymptotically Stable and Uniformly Ultimately Bounded Systems with Applications to Machine Vision

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***Abstract*—This paper addresses the stability of switching between a uniformly ultimately bounded (UUB) system and an asymptotically stable system with asymptotically decaying perturbation using multiple Lyapunov functions. It is proven that the switched system trajectories remain UUB if an average dwell time condition is satisfied, and the perturbation terms are bounded with a sufficiently small magnitude. The developed switched system stability results are applied to the state estimation of the perspective dynamical system in the presence of intermittent and biased velocity measurements using switched observers. Numerical simulations demonstrate the advantages of using the developed switched observer versus the individual observers.**

I. Introduction

Switched systems are generated by a switching signal from a collection of dynamical systems such that only a single subsystem is active at a particular instant. Switching between multiple dynamical systems can generate desirable behavior and trajectories, which might not be possible using a single subsystem. This technical note studies switching between an asymptotically stable subsystem and a UUB subsystem. Many systems exhibit asymptotic stability, for instance, adaptive control with parameter estimation, and uniformly ultimately bounded (UUB) stability, for example, systems with uncertainties and external disturbances.

It is well known that arbitrary switching between stable systems can lead to undesirable and unstable behaviors of the system [1]–[3]. The concept of average dwell time is developed in [4] to relax the conservative dwell time condition established from a stability analysis (e.g., a Lyapunov-based analysis),. If the average dwell time condition is satisfied, the switching is sufficiently slow to maintain the boundedness of the solution trajectories. Many switched system stability results are developed for linear systems, few of which are summarized next. A linear matrix inequality-based sufficient condition is developed in [5] for the analysis and control synthesis of discrete-time switched systems. An extension of LaSalle's invariance principle is provided in [6] for switched linear systems from multiple Lyapunov functions whose derivatives are only negative semi-definite. In [7], the stability results are surveyed for switched linear systems and the problem of stabilizability of switched systems is analyzed.

Lyapunov stability analysis tools based on multiple Lyapunov functions for nonlinear systems are developed in [1]. In [8] invariance-like results for nonautonomous nonlinear switched systems are presented. Switched and hybrid systems have proven to be useful in the analysis and design tools for output feedback control and state observers of nonlinear systems. In [9] a combined output feedback controller that switches between locally and globally asymptotically stable output feedback controllers is developed. A switching method is developed in [10] for combining local and global observers of nonlinear systems. The switching results developed so far for nonlinear systems, for example, [9], [11]–[13], require the individual systems to be exponentially stable, input-to-state stable (ISS) or input-output-to-state stable (IOSS). Many systems such as adaptive control or observer design only yield asymptotic stability. This technical note addresses switched system stability of two perturbed systems, an asymptotically stable system and a UUB system. The main contribution of the paper is to derive conditions under which the switched system remains stable. Average dwell time conditions are derived from a Lyapunov-based stability analysis. It is proved that the asymptotically stable subsystem enters the region of attraction of the UUB subsystem within a finite time. It is also proved that if the average dwell time condition is satisfied, then the switched system trajectories remain UUB. The switched system conditions are applied to a practical case of image-based depth observers when the camera velocities are available and when they are not for stable switching between two observers.

Switched systems-based analysis is popular in applications such as biped locomotion, image-based feedback systems that inherently involve multiple subsystems. For example, applications of switched system to biped locomotion are developed in [11], [14]. In [11] the boundedness of ISS stable switched systems with multiple equilibria is proven. For applications in image-feedback systems, an asymptotically stable visual servo controller is proposed in [15], which switches between image-based visual servo (IBVS) control and position-based visual servo control. A switched controller for switching between IBVS and dynamic movement primitives is proposed in [16]. A switched controller for active image-based depth estimation is proposed in [17]. Switched systems framework is used to tackle the issues of feature track losses, occlusions, and limited camera field of view for

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the image-based target tracking. In [18], [19] the switching between a state predictor and an image-based observer in the presence of intermittent image measurements is analyzed using a common Lyapunov function. In [20], the observer-predictor framework is used for trajectory tracking in the presence of intermittent state measurements. For image-based depth estimation application, although results exist that use full camera motion [21], [22] and part of the camera motion information for depth estimation [23], a switched observer framework can be useful in scenarios when such measurements are available intermittently due to a faulty sensor. The switched system analysis result developed in this paper is applied to the problem of depth estimation using switched observers when intermittent and biased velocity measurements are available.

Notations: The symbols $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the set of non-negative real numbers and non-negative integers, respectively. The open ball of radius δ around x is defined as $\mathcal{B}(x,\delta) = \left\{x' \in \mathbb{R}^p \mid \|x - x'\| < \delta\right\}$, where $\delta > 0$ is a constant. For a constant $\alpha > 0$ and a continuous nonnegative integrable function $\gamma(t)$, the shorthand $G_{t_0}^{t}(\alpha, \gamma(t)) = \int_{t_0}^{t} e^{-\alpha(t-\tau)}\gamma(\tau)\,d\tau$ and similarly $G_{t_0}^{t}(\gamma(t), \alpha) = \int_{t_0}^{t} e^{-\alpha\tau}\gamma(t-\tau)\,d\tau$ is used. The symbols $\rho_{\max}(\cdot)$ and $\rho_{\min}(\cdot)$ denote the maximum and minimum singular values of a matrix.

II. Problem Formulation

This section introduces the subsystems in the continuous-time switched system considered in this paper along with their stability results.

A. Family of Perturbed Subsystems

Consider the following family of continuous-time subsystems indexed by a finite set $\mathcal{P} = \{1, 2\}$ such that

$$\dot{x}(t) = f_p(x(t), t) + g_p(x(t), t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state at time t, and $p \in \mathcal{P}$ denotes the index of the active subsystem. The functions $f_1 : \mathcal{D}_1 \times \mathbb{R}_+ \to \mathbb{R}^n$ and $f_2 : \mathcal{D}_2 \times \mathbb{R}_+ \to \mathbb{R}^n$ are piecewise continuous in $t \in \mathbb{R}_+$ and locally Lipschitz on $x \in \mathcal{D}_1$ and $x \in \mathcal{D}_2$, respectively where $\mathcal{D}_1 \subset \mathcal{D}_2$. The terms $g_1 : \mathcal{D}_1 \times \mathbb{R}_+ \to \mathcal{G}_1$ and $g_2 : \mathcal{D}_2 \times \mathbb{R}_+ \to \mathcal{G}_2$ are perturbation terms, which are piecewise continuous in $t \in \mathbb{R}_+$ and locally Lipschitz in $x \in \mathcal{D}_1$ and $x \in \mathcal{D}_2$, respectively. The sets $\mathcal{G}_1$ and $\mathcal{G}_2$ are bounded sets. The perturbations have certain properties, which will lead to different stability results. In particular, the perturbation terms that satisfy the following bounds are considered, $\|g_1(x,t)\| \leq \alpha'\|x\| + \delta_1$, $\forall x \in \mathcal{D}_1$, $t \geq 0$, and $\|g_2(x,t)\| \leq \xi(t)$, $\forall x \in \mathcal{D}_2$, $t \geq 0$, where α', $\delta_1 > 0$ are constants, $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative, continuous, and bounded signal such that $\sup_{t\geq 0}\xi(t) \leq \delta_2$ for some $\delta_2 > 0$. Additionally, the function $\xi(t)$ satisfies the asymptotic property $\lim_{t\to\infty}\xi(t) = 0$.

B. Stability of Perturbed Subsystems

In this subsection, the stability properties of the individual perturbed systems are stated using the existing results from [24]. Let $x = 0$ be an equilibrium point of the nominal subsystems given by

$$\dot{x} = f_p(x,t), \quad p \in \mathcal{P}. \tag{2}$$

The following lemma establishes the stability of the perturbed dynamical system when the perturbation term is bounded by the term $\alpha'\|x\| + \delta_1$.

Lemma 1. *(Adapted from Lemma 9.2 of [24]) Let $x = 0$ be an exponentially stable equilibrium point of the nominal system in (2) for $p = 1$. Let $V_1(x,t) : \mathcal{D}_1 \times \mathbb{R}_+ \to \mathbb{R}_+$ be a $\mathcal{C}^1(\mathcal{D}_1, \mathbb{R}_+)$ Lyapunov function that satisfies*

$$\underline{c}_1\|x\|^2 \leq V_1(x,t) \leq \bar{c}_1\|x\|^2,$$
$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x}f_1(x,t) \leq -\underline{\lambda}_1'\|x\|^2, \quad \left\|\frac{\partial V_1}{\partial x}\right\| \leq \bar{\lambda}_1\|x\|, \tag{3}$$

$\forall\ (x,t) \in \mathcal{D}_1 \times \mathbb{R}_+$ for some $\underline{c}_1, \bar{c}_1, \bar{\lambda}_1 > 0$ and $\underline{\lambda}_1' > \bar{\lambda}_1\alpha'$, where $\mathcal{D}_1 \triangleq \mathcal{B}(0, r_1)$. Suppose the perturbation term bound in (1) satisfies $\delta_1 < \frac{\underline{\lambda}_1}{\bar{\lambda}_1}\sqrt{\frac{\underline{c}_1}{\bar{c}_1}}r_1$, $\forall t \geq 0$, $x \in \mathcal{D}_1$, where $\underline{\lambda}_1 \triangleq \underline{\lambda}_1' - \bar{\lambda}_1\alpha' > 0$, then $\forall\ \|x(t_0)\| < \sqrt{\underline{c}_1/\bar{c}_1}r_1$, the Lyapunov function in (3) of the perturbed system in (1) for $p = 1$ satisfies the bound

$$V_1(x(t), t) \leq V_1(x(t_0), t_0)\,e^{-\alpha_1(t-t_0)} + \frac{\bar{\lambda}_1^2}{2\underline{\lambda}_1}G_{t_0}^{t}\left(\alpha_1, \delta_1^2\right), \tag{4}$$

where $\alpha_1 = \underline{\lambda}_1/2\bar{c}_1$. The ultimate bound on $\|x(t)\|$ is given by $\limsup_{t\to\infty}\|x(t)\| \leq \sqrt{\frac{\bar{c}_1}{\underline{c}_1}}\frac{\bar{\lambda}_1}{\underline{\lambda}_1}\delta_1$.

Stability of the system in (1) when the perturbation term is bounded by a vanishing time varying function $\xi(t)$ is considered in the following lemma.

Lemma 2. *(Adapted from Lemma 9.5 case II of [24]) Let $x = 0$ be an exponentially stable equilibrium point of the nominal system (2) with $p = 2$. Let $V_2(x,t) : \mathcal{D}_2 \times \mathbb{R}_+ \to \mathbb{R}_+$ be a $\mathcal{C}^1(\mathcal{D}_2, \mathbb{R}_+)$ Lyapunov function that satisfies*

$$\underline{c}_2\|x\|^2 \leq V_2(x,t) \leq \bar{c}_2\|x\|^2,$$
$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x}f_2(x,t) \leq -\underline{\lambda}_2\|x\|^2, \quad \left\|\frac{\partial V_2}{\partial x}\right\| \leq \bar{\lambda}_2\|x\|, \tag{5}$$

$\forall\ (x,t) \in \mathcal{D}_2 \times \mathbb{R}_+$ for some $\underline{c}_2, \bar{c}_2, \underline{\lambda}_2, \bar{\lambda}_2 > 0$, where $\mathcal{D}_2 \triangleq \mathcal{B}(0, r_2)$. Suppose the perturbation term satisfies $\|g_2(x,t)\| \leq \xi(t)$, $\forall t \geq 0$, $x \in \mathcal{D}_2$, where $\xi(t)$ satisfies the asymptotic property, then $\forall\ \|x(t_0)\| < \sqrt{\underline{c}_2/\bar{c}_2}r_2$ and $\delta_2 < \frac{\underline{\lambda}_2}{\bar{\lambda}_2}\sqrt{\frac{\underline{c}_2}{\bar{c}_2}}r_2$, the Lyapunov function of the perturbed system (1) for $p = 2$ satisfies the bound given by

$$V_2(x(t), t) \leq V_2(x(t_0), t_0)e^{-\alpha_2(t-t_0)} + \frac{\bar{\lambda}_2^2}{2\underline{\lambda}_2}G_{t_0}^{t}\left(\alpha_2, \xi^2(t)\right) \tag{6}$$

where $\alpha_2 = \underline{\lambda}_2/2\bar{c}_2$, and the system is asymptotically stable in the sense that $\lim_{t\to\infty}\|x(t)\| = 0$. Furthermore, if all

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assumptions hold globally, then (6) and asymptotic stability are satisfied for any $x(t_0)$ and any bounded $\xi(t)$.

C. Switched System

Let $\sigma : \mathbb{R}^n \times \mathbb{R}_+ \to \mathcal{P}$ be a state and time dependent, right-continuous switching signal, which selects one of the subsystems in the finite set $\mathcal{P}$ to be active at time t, i.e., $\sigma(x,t) \in \mathcal{P}$. Consider the resulting switched system

$$\dot{x}(t) = f_{\sigma(x,t)}\left(x(t),t\right) + g_{\sigma(x,t)}\left(x(t),t\right), \tag{7}$$

whose solution $x(t) \triangleq \psi\left(t, x\left(t_0\right), \sigma\left(x\left(t\right), t\right)\right)$ is a concatenation of the solutions of the individual subsystems depending on the switching signal. Let $\{t_n\}_{n\in\mathbb{Z}_+}$ be a set of strictly increasing switching time instants. Owing to the continuity of $f_p\left(x,t\right)$ and $g_p\left(x,t\right)$ in (1), $x\left(t\right)$ is continuous between switching instances, i.e., the interval (t_n, t_{n+1}). For a switching time t_n, the active subsystem $f_{\sigma(x(t_n),t_n)}$ over the interval $[t_n, t_{n+1})$ has the initial condition $x\left(t_n\right) = \lim_{t\nearrow t_n} x\left(t\right)$, which establishes the continuity of $x\left(t\right)$ at t_n and thus for all $t \geq 0$.

III. Stability Results of the Switched System

In this section, the stability results of the switched system in (7) for a right continuous switching signal $\sigma\left(x,t\right)$ are presented. To facilitate the stability analysis, the following definitions are discussed.

Definition 1. *(Ch. 3, pp. 58 in [2]) The switching signal $\sigma(x,t)$ has average dwell time τ_D, if there exist two numbers $N_0 \in \mathbb{Z}_+$ and $\tau_D \in \mathbb{R}_+$ such that*

$$N_\sigma\left(t, \underline{t}\right) \leq N_0 + \frac{t - \underline{t}}{\tau_D} \tag{8}$$

is satisfied, where $N_0 \geq 1$ is known as the chatter bound and $N_\sigma(t, \underline{t})$ are the number of discontinuities on the interval $[\underline{t}, t)$.

Definition 2. *(Adapted from [11] and [25]) The system in (7) is practically stable for the perturbations $g_1 \in \mathcal{G}_1$, $g_2 \in \mathcal{G}_2$ and the switching signal $\sigma\left(x,t\right)$ with respect to the sets Ω_1 and Ω_2 such that $\Omega_1 \subset \Omega_2$ if $x\left(t_0\right) \in \Omega_1$ implies $x\left(t\right) \in \Omega_2, \ \forall\, t \geq t_0$.*

For analyzing the stability of the switched system using multiple Lyapunov functions, the following constants are defined $\underline{c} = \min_{p\in\mathcal{P}} \underline{c}_p$, $\bar{c} = \max_{p\in\mathcal{P}} \bar{c}_p$, $\mu = \frac{\bar{c}}{\underline{c}}$, where $\underline{c}_p$ and $\bar{c}_p$ are defined in (3) and (5). Additionally, define the constants $\underline{\lambda} = \min_{p\in\mathcal{P}} \underline{\lambda}_p$, $\bar{\lambda} = \max_{p\in\mathcal{P}} \bar{\lambda}_p$. From the definition of the above constants it is clear that $\min\{\alpha_1, \alpha_2\} \geq \underline{\lambda}/2\bar{c}$. In the following theorems, it will be shown that for sufficiently slow switching on average, the solution trajectories of the switched system $x\left(t\right) \triangleq \psi\left(t, x\left(t_0\right), \sigma\left(x\left(t\right), t\right)\right)$ in (7) remain bounded. The dwell time bound

$$\bar{\tau}_D = \frac{\ln \mu}{(\underline{\lambda}/2\bar{c}) - \epsilon}, \tag{9}$$

where $\epsilon \in \left(0, \frac{\underline{\lambda}}{2\bar{c}}\right)$, will be used as a lower bound for τ_D to ensure stable switching and boundedness of the solution trajectories of the switched system. The stability of the trajectories is subject to a hysteresis-based switching condition, which is designed to guarantee boundedness and eliminate the possibility of Zeno behavior, thereby averting a finite escape time. To prove the stability results, consider the following Lyapunov functions $V_p : \mathcal{D}_p \times \mathbb{R}_+ \to \mathbb{R}_+$ for which the following relations hold $V_p \leq \mu V_q, \ \forall\, t \geq 0, \ x \in \mathcal{D}_1, \ \forall\, p, \ q \in \mathcal{P}$, as $\mathcal{D}_1 = \mathcal{D}_1 \cap \mathcal{D}_2$ [2].

Theorem 1. *Consider the switched system in (7) with the assumption that there exist $\mathcal{C}^1\left(\mathcal{D}_p, \mathbb{R}_+\right)$ Lyapunov functions $V_p : \mathcal{D}_p \times \mathbb{R}_+ \to \mathbb{R}_+$ for each $p \in \mathcal{P}$ with the properties described in Lemma 1 and Lemma 2. Further assume that $\mathcal{D}_1 \subset \mathcal{D}_2$ and the following sufficient conditions hold*

1) *$\forall\ x\left(t\right) \in \mathcal{D}_2 \setminus \mathcal{D}_1$ the switching signal satisfies $\sigma\left(x\left(t\right), t\right) = 2$.*
2) *If $\sigma\left(x\left(t^-\right), t^-\right) = 2$, then $\sigma\left(x\left(t\right), t\right) = 1$ if $x\left(t\right) \in \mathcal{B}\left(0, \kappa\right)$ where $\kappa = \frac{r_1}{\sqrt{\mu^{N_0+2}}}$.*
3) *The constants r_1, r_2 and the perturbation bounds satisfy $r_2 > \sqrt{\frac{\bar{c}_2}{\underline{c}_2}} \max\left\{r_1, \sqrt{\bar{\delta}_1^2 + \bar{\delta}_2^2}\right\}, \ \ r_1 > \bar{\delta}_1,$*

where $\bar{\delta}_1 = \sqrt{\frac{\mu^{N_0+1}}{2\underline{\lambda}_1 \epsilon \underline{c}}}\bar{\lambda}_1 \delta_1$ and $\bar{\delta}_2 = \sqrt{\frac{\mu^{N_0+1}}{2\underline{\lambda}_2 \epsilon \underline{c}}}\bar{\lambda}_2 \delta_2$. Then $\exists\, \bar{\tau}_D > 0$ such that for the switching signal $\sigma\left(x,t\right)$ satisfying Conditions 1, 2 and the average dwell time constraint in (8) with $N_0 \geq 1$, $\tau_D \geq \bar{\tau}_D$, and $\forall\, x\left(0\right) \in \mathcal{B}\left(0, \sqrt{\underline{c}_2/\bar{c}_2}\, r_2\right)$ the following results hold.

1) *$\exists\ T_1\left(r_1, r_2, \bar{c}_2, \underline{c}_2, \underline{\lambda}_2, \varepsilon_1\right) \in \mathbb{R}_+$ for $\varepsilon_1 > 0$ such that $x\left(T_1\right) \in \mathcal{D}_1$.*
2) *$\exists\, T_2\left(\varepsilon_2\right) \geq T_1$ for $\varepsilon_2 > 0$ such that the switched system (7) is practically stable with respect to $\Omega_1 \triangleq \mathcal{B}\left(0, \kappa\right)$ and $\Omega_2 \triangleq \mathcal{D}_1$ such that $\forall\, t \geq T_2$, the solutions $x\left(t\right) \triangleq \psi\left(t, x\left(0\right), \sigma\left(x(t), t\right)\right) \in \mathcal{D}_1$.*
3) *The trajectories remain uniformly ultimately bounded in the sense that $\limsup_{t\to\infty} \|x\left(t\right)\| \leq \bar{\delta}_1$.*

Furthermore, if $\mathcal{D}_2 = \mathbb{R}^n$, then Condition 3 and the Results 1 - 3 hold for any $x\left(0\right)$ and any bounded $\xi\left(t\right)$.

Proof: The proof is divided into three parts.
Result 1: Consider the Lyapunov function $V_p : \mathcal{D}_p \times \mathbb{R}_+ \to \mathbb{R}_+$. If $x\left(0\right) \in \mathcal{D}_1$, then Result 1 holds trivially for $T_1 = 0$. Using the definition of the limit, there exists a $\mathcal{T}_1\left(\varepsilon_1\right)$ for every $\varepsilon_1 \in \left(0, \delta_2\right)$ such that $\forall t \geq \mathcal{T}_1\left(\varepsilon_1\right), \xi\left(t\right) \leq \varepsilon_1$ which implies $\sup_{t\geq\mathcal{T}_1} \xi\left(t\right) \leq \varepsilon_1$. If $x\left(\mathcal{T}_1(\varepsilon_1)\right) \in \mathcal{B}\left(0, \sqrt{\underline{c}_2/\bar{c}_2}\, r_2\right) \setminus \mathcal{D}_1$ and Condition 1 is satisfied for $t \in [\mathcal{T}_1(\varepsilon_1), T_1)$ that is $\sigma : \mathbb{R}^n \times [\mathcal{T}_1(\varepsilon_1), T_1) \to 2$ for some $T_1 > \mathcal{T}_1(\varepsilon_1)$, then $V_{\sigma(x,t)}$ is differentiable on the interval $[\mathcal{T}_1(\varepsilon), T_1)$, and the derivative of the Lyapunov function satisfies

$$\begin{aligned}\dot{V}_{\sigma(x,t)} &\leq -\underline{\lambda}_2\|x\|^2 + \xi\left(t\right)\bar{\lambda}_2\|x\| \leq -\underline{\lambda}_2\|x\|^2 + \varepsilon_1\bar{\lambda}_2\|x\| \\ &\leq -\frac{\underline{\lambda}_2}{2\bar{c}_2}V_{\sigma(x,t)}, \ \forall\|x\| \geq \frac{2\varepsilon_1\bar{\lambda}_2}{\underline{\lambda}_2}, \ t \in [\mathcal{T}_1(\varepsilon_1), T_1)\end{aligned} \tag{10}$$

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which leads to the bound $\|x(t)\| \leq \sqrt{\frac{\bar{c}_2}{\underline{c}_2}} r_2 e^{-\frac{\underline{\lambda}_2}{4\bar{c}_2}(t-\mathcal{T}_1(\varepsilon_1))}$, $\forall\ \|x\| \geq \frac{2\varepsilon_1 \bar{\lambda}_2}{\underline{\lambda}_2}$, $t \in [\mathcal{T}_1(\varepsilon_1), T_1)$. Choosing $\varepsilon_1 \in \left(0, \min\left\{\underline{\lambda}_2 r_1/2\bar{\lambda}_2, \delta_2\right\}\right)$, it is sufficient to pick $T_1(r_1, r_2, \bar{c}_2, \underline{c}_2, \underline{\lambda}_2, \varepsilon_1) \in \left(\mathcal{T}_1(\varepsilon_1) + \frac{4\bar{c}_2}{\underline{\lambda}_2}\ln\left(\sqrt{\frac{\bar{c}_2}{\underline{c}_2}}\frac{r_2}{r_1}\right), \infty\right)$ ensuring $x(T_1) \in \mathcal{D}_1$.

Result 2: Let $N_\sigma(t, \underline{t})$ be the number of switching instants on the interval $t \in [\underline{t}, t)$. Also, let $\underline{t} = \inf\left\{t \in \mathbb{R}_+ \middle| \|x(t)\| < \kappa\right\}$ and let $\{t_n\}_{n=0}^{N_\sigma}$ be a strictly monotonically increasing sequence of switching times with $t_0 = \underline{t}$. The existence of such a $\underline{t}$ can be established using a similar argument as that of Result 1. Next, a recursion of upper bounds in (4) and (6) using μ is followed to compute an upper bound on $V_{\sigma(x,t)}(x,t)$. Without loss of generality, assume that the Subsystem 1 is active on the interval $[t_{N_\sigma}, t)$, then the following upper bound is obtained

$$\begin{aligned}
&V_\sigma(x(t),t) \leq \mu^{N_\sigma} V_\sigma(x(\underline{t}),\underline{t})\, e^{-\frac{\underline{\lambda}}{2\bar{c}}(t-\underline{t})} \\
&+ \frac{\bar{\lambda}_1^2}{2\underline{\lambda}_1}\sum_{k=0}^{N_\sigma(t,\underline{t})} \mu^{N_\sigma-k} e^{-\frac{\underline{\lambda}}{2\bar{c}}(t-t_{k+1})} G_{t_k}^{t_{k+1}}\left(\alpha_1, \delta_1^2\right) \\
&+ \frac{\bar{\lambda}_2^2}{2\underline{\lambda}_2}\sum_{k=0}^{N_\sigma(t,\underline{t})} \mu^{N_\sigma-k} e^{-\frac{\underline{\lambda}}{2\bar{c}}(t-t_{k+1})} G_{t_k}^{t_{k+1}}\left(\alpha_2, \xi^2(t)\right) \qquad (11)
\end{aligned}$$

$\forall\ x \in \mathcal{D}_1$, where the arguments of σ are dropped for brevity. Using the fact that $G_{t_k}^{t_{k+1}}(\min\{\alpha_1,\alpha_2\},\cdot) \leq G_{t_k}^{t_{k+1}}(\underline{\lambda}/2\bar{c},\cdot) \leq G_{t_k}^{t_{k+1}}(\epsilon,\cdot)$ for any positive function or constant, the fact $N_\sigma(t,\underline{t}) - k - 1 \leq N_\sigma(t, t_{k+1})$, and by substituting (9), (11) can be simplified and upper bounded as

$$\begin{aligned}
V_\sigma(x(t),t) \leq \mu^{N_0+1}\Bigg(&V_\sigma(x(\underline{t}),\underline{t})\, e^{-\epsilon(t-\underline{t})} + \frac{\bar{\lambda}_1^2}{2\underline{\lambda}_1} G_{\underline{t}}^{t}\left(\epsilon, \delta_1^2\right) \\
&+ \frac{\bar{\lambda}_2^2}{2\underline{\lambda}_2} G_{\underline{t}}^{t}\left(\epsilon, \xi^2(t)\right)\Bigg). \qquad (12)
\end{aligned}$$

From (12) it can be concluded that $\|x(t)\| \leq \max\left\{\frac{r_1}{\kappa}\|x(\underline{t})\|, \sqrt{\bar{\delta}_1^2 + \bar{\delta}_2^2}\right\} \leq \sqrt{\underline{c}_2/\bar{c}_2}\, r_2$. Now by the definition of the limit, there exists a $\mathcal{T}_2(\varepsilon_2)$ such that, $\forall t \geq \mathcal{T}_2(\varepsilon_2)$, $\xi^2(t) \leq \varepsilon_2$ for every $\varepsilon_2 \in (0, \delta_2^2)$, which implies that $\sup_{t \geq \mathcal{T}_2} \xi^2(t) \leq \varepsilon_2$. Pick $\varepsilon_2 \in \left(0, \min\left\{\sqrt{\frac{2\underline{\lambda}_2 c\epsilon}{\mu^{N_0+1}\bar{\lambda}_2^2}\left(r_1^2 - \bar{\delta}_1^2\right)}, \delta_2^2\right\}\right)$ which leads to $\|x(t)\| < r_1$ if $\|x(\mathcal{T}_2)\| < \kappa$, $\forall\ t \geq \mathcal{T}_2(\varepsilon_2)$. Such an $\varepsilon_2 > 0$ exists from Condition 3. Choose $T_2(\varepsilon_2) = \inf\left\{t \in \mathbb{R}_+ \middle| t \geq \mathcal{T}_2(\varepsilon_2), \|x(t)\| < \kappa\right\}$, which ensures the practical stability of the switched system with respect to Ω_1 and Ω_2, $\forall\, t \geq T_2(\varepsilon_2)$.

Result 3: Consider (12) for $t \in (T_2(\varepsilon_2), \infty)$. Evaluating the term $\frac{\bar{\lambda}_1^2}{2\underline{\lambda}_1} G_{T_2}^{t}\left(\epsilon, \delta_1^2\right) = \frac{\bar{\lambda}_1^2 \delta_1^2}{2\underline{\lambda}_1 \epsilon}\left(1 - e^{-\epsilon(t - T_2(\varepsilon_2))}\right)$, taking limit superior on both sides, using the reverse version of Fatou's lemma and the Lebesgue dominated convergence theorem [26] to obtain $\limsup_{t\to\infty} G_{T_2}^{t}\left(\epsilon, \xi^2(t)\right) \leq G_{T_2}^{t}\left(\limsup_{t\to\infty}\left(\epsilon, \xi^2(t)\right)\right) = 0$. Result 3 is obtained using the Lyapunov function bounds for switched system. ∎

Remark 1. *The result of Theorem 1 does not establish the invariance of $\mathcal{D}_1$ with respect to the switched system trajectories. In general for $t \in [0, T_2)$, the system trajectories may exit $\mathcal{D}_1$ but are always in the interior of $\mathcal{D}_2$ given that the average dwell time condition in (9) and, Conditions 1-3 of Theorem 1 are satisfied. If the trajectories exit $\mathcal{D}_1$, Subsystem 2 becomes active due to which the vector field is pointing inwards, forcing the trajectory back into $\mathcal{D}_1$. After time T_2 the perturbation term $g_2(x,t)$ becomes small enough so that the trajectories never exit $\mathcal{D}_1$. Note that, T_2 can be shortened by picking appropriate $\underline{\lambda}_2$.*

IV. Application to Observer Design for Perspective Dynamical System

Consider a camera in motion with linear and angular velocities viewing feature points on a static object. Let $\bar{m}(t) \triangleq [X(t),\, Y(t),\, Z(t)]^\top \in \mathcal{X}$ be the Euclidean coordinates of a feature point belonging to the static object seen by the camera in the camera reference frame. The set $\mathcal{X} \subseteq \mathbb{R}^3$ is bounded and closed. Consider the state vector $z(t) \triangleq [y_1(t),\, y_2(t),\, y_3(t)]^\top \in \mathcal{Y}$ such that $\mathcal{Y} \subseteq \mathbb{R}^3$ is a closed and bounded set, where $y_1(t) = X(t)/Z(t)$, $y_2(t) = Y(t)/Z(t)$, $y_3(t) = 1/Z(t)$.

Remark 2. *The image plane coordinates $y_1(t)$ and $y_2(t)$ are bounded by known constants $\underline{y_1} \leq y_1(t) \leq \overline{y_1}$ and $\underline{y_2} \leq y_2(t) \leq \overline{y_2}$ due to the image size. The inverse depth $y_3(t)$ can be lower and upper bounded using the known constants $0 < \underline{y_3} < y_3(t) \leq \overline{y_3}$ [27].*

Assumption 1. *The depth of the feature point $Z(t)$ is invertible in the compact set $\mathcal{Y}$.*

The state dynamics and the measurement model can be written as

$$\dot{z}(t) = F(z(t), u(t)) = \begin{bmatrix} f_m(s)\,\omega + \Omega^\top(s,v)\, y_3 \\ f_u(s, y_3, v, \omega) \end{bmatrix} \qquad (13)$$

$$s(t) = Hz(t) \qquad (14)$$

where $H = \begin{bmatrix} I_2 & 0_{2\times 1} \end{bmatrix}$ is the measurement matrix and $s(t) \in \mathcal{S}$ are the measurements of the dynamical system such that $\mathcal{S} \triangleq \left\{s \in \mathbb{R}^2 \middle| \underline{y_1} \leq y_1(t) \leq \overline{y_1}, \underline{y_2} \leq y_2(t) \leq \overline{y_2}\right\}$. The velocity vector $u(t) = \left[v^\top(t)\ \omega^\top(t)\right]^\top \in \mathcal{U}$, $v(t) = [v_X(t)\ v_Y(t)\ v_Z(t)]^\top \in \mathcal{V}$ are the linear velocities and $\omega(t) = [\omega_X(t)\ \omega_Y(t)\ \omega_Z(t)]^\top \in \mathcal{W}$ are the angular velocities of the camera in the camera body reference frame. The sets $\mathcal{U} \subset \mathbb{R}^6$, $\mathcal{V} \subset \mathbb{R}^3$ and $\mathcal{W} \subset \mathbb{R}^3$ are bounded. The functions, $f_m(s) \in \mathbb{R}^2$ and $\Omega(s,v) \in \mathbb{R}^{1\times 2}$ are defined by

$$\begin{aligned}
f_m(s) &= \begin{bmatrix} y_1 y_2 & -\left(1 + y_1^2\right) & y_2 \\ 1 + y_2^2 & -y_1 y_2 & -y_1 \end{bmatrix} \\
\Omega(s,v) &= \begin{bmatrix} y_1 v_Z - v_X & y_2 v_Z - v_Y \end{bmatrix} \\
f_u(s, y_3, v, \omega) &= v_Z y_3^2 + (y_2\omega_X - y_1\omega_Y)\, y_3 \qquad (15)
\end{aligned}$$

Assumption 2. *The velocities $u(t)$ can be measured intermittently. The intermittent biased measurements of the velocities are of the form $u_d(t) = u(t) + d(t)$, where $d(t)$*

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is an unknown disturbance with $\sup_{t\geq 0}\|d(t)\| \leq \bar{d}$*, where* $\bar{d} > 0$.

Remark 3. *The intermittent availability of the biased velocity measurements can be attributed to a faulty motion sensor such as the inertial measurement unit (IMU).*

Assumption 3. *The functions* $f_m(s)$, $\Omega(s,v)$, *and* $f_u(s, y_3, v, \omega)$ *are bounded and the observability condition* $\inf_{t\geq 0}\Omega(s(t), v(t))\Omega^\top(s(t), v(t)) > k_1$ *is satisfied, where* $k_1 > 0$ *[28].*

For further development of the observers and respective stability analysis, the state estimation error is defined as $e(t) = z(t) - \hat{z}(t)$, where $\hat{z}(t)$ is the state estimate.

Problem Definition and Solution Approach. Given the feature point measurements $s(t)$, and the intermittent and biased measurements of the velocity $u_d(t)$, the goal is to estimate the state $z(t)$ such that $\|e(t)\| \leq \bar{\delta}_1$ as $t \to \infty$. A switching-based observer is developed to address this problem, where an EKF observer is used, which yields locally bounded estimation error when the biased velocity measurement is available and an observer, which yields asymptotically stable estimation error, when the velocity measurement is not available.

A. Locally Bounded Observer

In this subsection, a full order nonlinear observer is described, which is activated when the biased velocity measurements are available. In particular the Extended Kalman Filter (EKF) is used as a deterministic nonlinear local observer. Consider the EKF as a nonlinear observer described by the following equations [29]

$$\dot{\hat{z}}(t) = F(\hat{z}(t), u_d(t)) + P(t)H^\top R^{-1}(s(t) - H\hat{z}(t)) \tag{16}$$

$$\dot{P}(t) = \left(A^\top(t) + \bar{\alpha}I_3\right)P(t) + P(t)\left(A(t) + \bar{\alpha}I_3\right) + W - P(t)H^\top R^{-1}HP(t) \tag{17}$$

where $A(t) = \frac{\partial F}{\partial z}\big|_{z=\hat{z}(t), u_d(t)}$ and W, R are symmetric positive definite matrices of appropriate dimensions, $P(t_0) = P_0 > 0$ and $\bar{\alpha} > 0$. The modified differential Riccati equation in (17) can be solved together with the state estimator in (16) using numerical integration method such as Runge-Kutta method [29], [30].

Assumption 4. *The solution* $P(t)$ *of (17) exists for all* $t \geq 0$ *and satisfies* $\underline{p}I_3 = \inf_{t\geq 0}\|P^{-1}(t)\|I_3 \leq P^{-1}(t) \leq \sup_{t\geq 0}\|P^{-1}(t)\|I_3 = \bar{p}I_3$.

The error dynamics after a Taylor expansion around $\hat{z}(t)$ and $u_d(t)$ can be written as

$$\begin{aligned}\dot{e} &= \left(A - PH^\top R^{-1}H\right)e + \Delta(\hat{z}, z, u, u_d) + Q(\hat{s}, \hat{y}_3)d \\ &= f_1(e(t), t) + g_1(t)\end{aligned} \tag{18}$$

where $\Delta(\hat{z}, z, u, u_d) = \Delta_1(\hat{z}, z, u, u_d) + \Delta_2(\hat{z}, z, u, u_d)$ is a function due to the higher order terms. The matrix $Q(\hat{s}, \hat{y}_3)$ in (18) is defined as

$$Q(\hat{s}, \hat{y}_3) = \begin{bmatrix} -\hat{y}_3 & 0 & \hat{y}_1\hat{y}_3 & \hat{y}_1\hat{y}_2 & -\left(1+\hat{y}_1^2\right) & \hat{y}_2 \\ 0 & -\hat{y}_3 & \hat{y}_2\hat{y}_3 & 1+\hat{y}_2^2 & -\hat{y}_1\hat{y}_2 & -\hat{y}_1 \\ 0 & 0 & \hat{y}_3^2 & \hat{y}_2\hat{y}_3 & -\hat{y}_1\hat{y}_3 & 0 \end{bmatrix}. \tag{19}$$

The separation of Δ into Δ_1 and Δ_2 is based on the fact that F is differentiable with respect to z at most twice almost everywhere (a.e.) and only once a.e. with respect to u for the particular case of the PDS. The function Δ_1 contains terms from $\frac{\partial F_i}{\partial z_j \partial z_k}e_j e_k$ and $\frac{\partial F_i}{\partial z_j \partial z_k \partial u_l}e_j e_k d_l$ and Δ_2 contains the terms from $\frac{\partial F_i}{\partial z_j \partial u_l}e_j d_l$, where $\forall\, i, j, k = 1, \cdots, 3$ and $\forall\, l = 1, \cdots, 6$. Let $f_1(e,t) = \left(A(t) - P(t)H^\top R^{-1}H\right)e(t) + \Delta_1(\hat{z}, z, u, u_d)$ and let $g_1(t) = \Delta_2(\hat{z}, z, u, u_d) + Q(\hat{s}, \hat{y}_3)d$. The terms for the case of the PDS are bounded as $\Delta_1(\hat{z}, z, u, u_d) \leq \bar{\Delta}_1\|e\|^2 + \bar{\Delta}_1'\|e\|^2\|d\| \leq \left(\bar{\Delta}_1 + \bar{\Delta}_1'\bar{d}\right)\|e\|^2$ and $\|\Delta_2(\hat{z}, z, u, u_d) + Q(\hat{s}, \hat{y}_3)d\| \leq \bar{\Delta}_2\|e\|\|d\| + \upsilon\|d\| \leq \bar{\Delta}_2\bar{d}\|e\| + \upsilon\bar{d}$, $\forall\, \|e\| \leq r_1', \|d\| \leq \bar{d}, u \in \mathcal{U}$ for appropriate $r_1', \bar{d} > 0$, implying that $\hat{z} \in \mathcal{B}(z, r_1')$, where $\bar{\Delta}_1, \bar{\Delta}_2$, $r_1' > 0$ and $\upsilon \triangleq \sup_{\hat{z}\in\mathcal{B}(z, r_1')}\rho_{\max}(Q(\hat{s}, \hat{y}_3))$. Consider the Lyapunov function $V_1 : \mathcal{D}_1 \times \mathbb{R}_+ \to \mathbb{R}_+$ defined as

$$V_1(e,t) = e^\top P^{-1}(t)e \tag{20}$$

where $\mathcal{D}_1 \triangleq \mathcal{B}(0, r_1)$ such that $r_1 = \min\left\{r_1', \frac{\lambda_{\min}(W)\underline{p}^2}{4\left(\bar{\Delta}_1 + \bar{\Delta}_1'\bar{d}\right)\bar{p}}\right\}$. Given the form of the Lyapunov function it is clear that $\underline{c}_1 = \underline{p}$, $\bar{c}_1 = \bar{p}$, $\delta_1 = \upsilon\bar{d}$ and $\bar{\lambda}_1 = 2\bar{p}$. In the next lemma, it is proved that the estimation error of the EKF remains bounded.

Lemma 3. *Suppose the disturbance term satisfies* $\bar{d} < \sqrt{\frac{\underline{p}}{\bar{p}}}\frac{\underline{\lambda}_1 r_1}{2\bar{p}\upsilon}$*, and the adjustable parameter is chosen according to the sufficient condition* $\bar{\alpha} > \max\left\{0, \frac{4\bar{p}\bar{\Delta}_2\bar{d} - \underline{p}^2\rho_{\min}(W)}{4\underline{p}}\right\}$ *then* $\forall\, \|e(t_0)\| < \sqrt{\frac{\underline{p}}{\bar{p}}}r_1$*, the Lyapunov function in (20) satisfies the bound*

$$V_1(e(t), t) \leq V_1(e(t_0), t_0)e^{-\alpha_1(t-t_0)} + \frac{2\bar{p}^2}{\underline{\lambda}_1}G_{t_0}^t\left(\alpha_1, \delta_1^2\right) \tag{21}$$

where $\alpha_1 = \underline{\lambda}_1/2\bar{p}$*. Furthermore, the ultimate bound on* $\|e(t)\|$ *is given by* $\limsup_{t\to\infty}\|e(t)\| \leq \sqrt{\frac{\bar{p}}{\underline{p}}}\frac{2\bar{p}}{\underline{\lambda}_1}\delta_1$.

Proof: Under Assumptions 3 and 4, the Lyapunov function in (20) satisfies

$$\begin{gathered}\underline{p}\|e\|^2 \leq V_1(e,t) \leq \bar{p}\|e\|^2, \\ \frac{dV_1}{de}f_1(e,t) \leq -\underline{\lambda}_1'\|e\|^2, \quad \left\|\frac{dV_1}{de}\right\| \leq 2\bar{p}\|e\|,\end{gathered} \tag{22}$$

$\forall\, e \in \mathcal{D}_1$, where $\underline{\lambda}_1' = 0.5\underline{p}^2\rho_{\min}(W) + 2\bar{\alpha}\underline{p} > 0$ [29]. Then using Lemma 1, $\underline{\lambda}_1 \triangleq \underline{\lambda}_1' - 2\bar{p}\bar{\Delta}_2\bar{d} > 0$ and it is required that $\delta_1 < \sqrt{\frac{\underline{p}}{\bar{p}}}\frac{\underline{\lambda}_1}{2\bar{p}}r_1$. Given the expression of δ_1, the bound on $\bar{d}$ can be obtained. Then (21) and the ultimate bound follow from Lemma 1. ■

B. Asymptotically Stable Observer

In this subsection, a full order nonlinear observer is described, which is activated when the velocity measurements are unavailable. Consider an observer of the form

$$\dot{\hat{z}} = \begin{bmatrix} f_m(s)\hat{\omega} + \Omega^\top(s,\hat{v})\hat{y}_3 + \Gamma(s - H\hat{z}) \\ f_u(s,\hat{y}_3,\hat{v},\hat{\omega}) + k_2\Omega(s,\hat{v})(s - H\hat{z}) \end{bmatrix} \tag{23}$$

where $\Gamma \in \mathbb{R}^{2\times 2}$ is symmetric, $\Gamma > 0$ and $k_2 > 0$ are suitable observer gains. The estimate of the velocity $\hat{u}(t)$ is obtained using the method in [31]. It is assumed that four image feature points, which can be tracked across frames, are available for the velocity estimator along with the knowledge of the initial rotation between the camera and object frame and constant coordinates of one feature point in the object frame. The estimation error of the velocity is defined by $\tilde{u}(t) = u(t) - \hat{u}(t)$. The velocity observer in [31] guarantees that the velocity estimation error remains bounded and the velocity is identified asymptotically, i.e., $\|\tilde{u}(t)\| \to 0$ as $t \to \infty$. Using the definition of the velocity estimation error, the observer error dynamics can be written as

$$\dot{e} = f_2(e,t) + g_2(e,\tilde{u},t) \tag{24}$$

where $e(t) \in \mathbb{R}^3$ and

$$f_2 = \begin{bmatrix} -\Gamma & \Omega^\top(s,v) \\ -k_2\Omega(s,v) & 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ \tilde{f}_u(e,t) \end{bmatrix},$$
$$g_2 = (Q(s,\hat{y}_3) + k_2 B(e,s))\,\tilde{u}(t) \tag{25}$$

such that

$$B(e,s) = \begin{bmatrix} & 0_{2\times 6} & & \\ -e_1 & -e_2 & y_1e_1 + y_2e_2 & 0_{1\times 3} \end{bmatrix}, \tag{26}$$

$e_1(t) = y_1(t) - \hat{y}_1(t)$, $e_2(t) = y_2(t) - \hat{y}_2(t)$ and $\tilde{f}_u(e,t) = f_u(s,y_3,v,\omega) - f_u(s,\hat{y}_3,v,\omega)$ and $\|\tilde{f}(e,t)\| \le L_{\tilde{f}}\|y_3 - \hat{y}_3\|$ for the Lipschitz constant $L_{\tilde{f}} > 0$ when $\hat{y}_3(t)$ is bounded.

Remark 4. *The estimate $\hat{z}(t)$ of (23) can be projected on the convex hypercube $\mathcal{Z} \triangleq \{\hat{z} \in \mathbb{R}^3 | \underline{y_i} - \iota \le \hat{z}_i \le \overline{y_i} + \iota,\ i = 1,2,3\}$ using the Lipschitz projection law in [22] where $\iota > 0$.*

Remark 5. *Since $s(t)$, $v(t)$ are bounded, the perturbation term can be bounded as $\|g_2(e,t)\| \le \delta_2'\|\tilde{u}(t)\| = \xi(t)$ with $\delta_2' \triangleq \sup_{e\in\mathcal{D}_2, \hat{z}\in\mathcal{B}(z,r_2)} \rho_{\max}(Q(s,\hat{y}_3) + B(e,s)) > 0$, $\xi(t) \to 0$ as $t \to \infty$ and $\delta_2 \triangleq \delta_2' \sup_{t\ge 0}\|\tilde{u}(t)\|$, where $\mathcal{D}_2 \triangleq \mathcal{B}(0,r_2)$ is a domain.*

Consider the Lyapunov function $V_2 : \mathcal{D}_2 \to \mathbb{R}_+$, with an arbitrarily large r_2 defined as

$$V_2(e) = \frac{1}{2}e^\top e, \tag{27}$$

where $e(0)$ is contained in $\mathcal{D}_2$. Given the form of the Lyapunov function it is clear that $\underline{c}_2 = \bar{c}_2 = 1/2$ and $\bar{\lambda}_2 = 1$. In the next lemma, asymptotic stability of the error dynamics in (24) using the Lyapunov function in (27) is proven.

Lemma 4. *Under Assumption 3, the Lyapunov function in (27) satisfies*

$$\frac{dV_2}{de} f_2(e,t) \le -\underline{\lambda}_2\|e\|^2, \tag{28}$$

$\forall e(t) \in \mathcal{D}_2$, where $\underline{\lambda}_2 = \min\left\{\rho_{\min}(\Gamma), \frac{(1-k_2)^2}{\rho_{\max}(\Gamma)}k_1 - L_{\tilde{f}}\right\} > 0$. If the velocity estimation error satisfies the sufficient condition

$$\sup_{t\ge 0}\|\tilde{u}(t)\| < \frac{\underline{\lambda}_2 r_2}{\delta_2'} \tag{29}$$

then $\forall\, \|e(t_0)\| < r_2$ the Lyapunov function in (27) satisfies the bound

$$V_2(e(t)) \le V_2(e(t_0))e^{-\alpha_2(t-t_0)} + \frac{1}{2\underline{\lambda}_2} G_{t_0}^t\left(\alpha_2, \xi^2(t)\right) \tag{30}$$

where $\alpha_2 = \underline{\lambda}_2$ and the system is asymptotically stable in the sense that $\lim_{t\to\infty}\|e(t)\| = 0$.

Proof: The proof of (28) follows from [21] and by applying the Schur Complement Lemma. The results in (30) and the asymptotic stability follow from Lemma 2. ∎

Remark 6. *The conditions of Lemma 4 can be satisfied by initializing $\hat{u}(0)$ sufficiently close to $u(0)$.*

C. Switching between UUB and Asymptotically Stable Observers

In this subsection the switching between the UUB observer of Section IV-A and the asymptotically stable observer of Section IV-B is discussed based on the framework outlined in Section II-C when intermittent and biased velocity measurements are available. Consider the switched error dynamics generated by switching between the error dynamics in (18) and (24) of the UUB observer and the asymptotically stable observer, respectively, and denoted by

$$\dot{e} = f_{\sigma(e,t)}(e,t) + g_{\sigma(e,t)}(e,t), \tag{31}$$

where $\sigma : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathcal{P}$ is a suitable switching signal with $\mathcal{P} = \{1,2\}$. The corresponding stability result is described first based on the result presented in Section III. Later a switching strategy is discussed for the two observers. To facilitate the analysis, consider the following constants $\underline{c} = \max\{\underline{p}, \frac{1}{2}\}$, $\bar{c} = \max\{\bar{p}, \frac{1}{2}\}$, $\mu = \frac{\bar{c}}{\underline{c}}$, $\underline{\lambda} = \min\{\underline{\lambda}_1, \underline{\lambda}_2\}$, $\bar{\lambda} = \max\{2\bar{p}, 1\}$, where $\underline{\lambda}_1$ and $\underline{\lambda}_2$ are defined according to the convergence rates of the UUB and the asymptotically stable observers defined according to Lemma 3 and Lemma 4, respectively. The following corollary establishes the stability of the switched observer error system in (31) when the average dwell time condition is satisfied.

Corollary 1. *Let the switched observer error dynamics in (31) satisfy the conditions of Lemma 1-4 and Theorem 1 along with Assumption 4. Then the Results 1 and 2 of Theorem 1 hold for some ε_1, $\varepsilon_2 > 0$ if the average dwell*

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time condition in (8) is satisfied. Additionally, the estimation error is uniformly ultimately bounded in the sense that

$$\limsup_{t\to\infty} \|e(t)\| \leq \sqrt{\frac{2\mu^{N_0+1}}{\underline{\lambda}\epsilon\bar{c}}}\bar{p}\upsilon\bar{d}. \tag{32}$$

Proof: The proof follows the proof of Theorem 1. ■

The corollary states that the estimation error remains bounded with an ultimate bound, which is proportional to the disturbance bound $\bar{d}$ under a suitable switching signal.

Remark 7. *Suppose the velocity measurements are available, then the EKF in the switched observer can be used. The EKF is robust to measurement noise [30], but can only be used when the velocity measurements are available. When the velocity measurements are not directly available from the motion sensor, the asymptotically stable observer in Section IV-B can be used to estimate the system state by estimating the velocities using camera images. However, in practice velocity estimation requires the computation and decomposition of the homography matrix from noisy feature points, which can affect the velocity estimation. In addition, the velocity estimation errors affect the convergence bound of the asymptotically stable observer, i.e.,* $\delta_3' \in \left(\frac{\sup_{t\geq t'}\xi^2(t)}{2\underline{\lambda}_2^2}, V_2(t')\right]$, *for some time* t' *as seen from the Lyapunov analysis. Thus, the switched observer is a preferred strategy, which uses EKF when the velocities can be measured from the sensor and asymptotically stable observer when the velocities are not available. The switched observer is also stable over a larger domain than the EKF.*

Remark 8. *The EKF cannot be used in its original form of Section IV-A in the absence of the velocity measurements. If it is used with a zero order hold (ZOH) for velocities, the state estimates become unstable as demonstrated in simulations.*

D. Switching Strategy

In this subsection, a switching strategy is developed to switch between the observers in Section IV-A and Section IV-B. The three criteria considered for switching between the observers are:

- The norm of the estimation error is less than a given bound, i.e., $\|e(t)\| < \kappa$.
- The availability of the biased velocity measurements i.e., $u_d(t)$.
- Satisfaction of the average dwell time constraint $\bar{\tau}_D$.

The first condition implies that the local observer, i.e., the EKF, should only be activated when the norm of the estimation error is sufficiently small as per Condition 2 of Theorem 1. However, the norm of the estimation error cannot be computed directly. An alternative is to estimate the norm of the estimation error from output data [10], [32], [33] if $\sup_{t\geq\bar{t}}\|\tilde{u}(t)\|$ is sufficiently small for some suitable $\bar{t} > 0$. A norm estimator of the form $\dot{b} = -\underline{\lambda}_2 b + k_b\|s-\hat{s}\|^2$ with $b(0) \in \mathbb{R}_+$, which approximates V_2, is used to detect when $\|e\| \leq \kappa'$, where $\kappa' \in (0,\kappa)$. It can be shown that if the velocities are available, the first switch to EKF after starting with the global observer occurs in finite time. Let $\bar{t} \triangleq \inf\{t' \in \mathbb{R}_+ | \xi(t) < \varepsilon', \forall t \geq t'\}$, where $\varepsilon' \in \left(0, \frac{\underline{\lambda}_2\kappa'}{\sqrt{2}}\right]$. For $k_b \in \left(0, \frac{\underline{\lambda}_2}{2}\right]$, $\varpi \in \left(\frac{\varepsilon'^2}{2\underline{\lambda}_2^2}, \frac{\kappa'^2}{2} - \frac{\varepsilon'^2}{2\underline{\lambda}_2^2}\right)$ and $c' \in \left(\varpi, \frac{\kappa'^2}{2} - \frac{\varepsilon'^2}{2\underline{\lambda}_2^2}\right]$, the switch occurs for $t \geq \bar{t}' + \frac{1}{\underline{\lambda}_2}\ln\left(\frac{b_{\bar{t}'}}{c'-\varpi}\right)$, where $b_{\bar{t}'} \triangleq b(\bar{t}') \in \mathbb{R}_+$, such that $V_2(e(t)) \leq \varpi$, $\forall t \geq \bar{t}' \geq \bar{t}$ and the detection condition is $b(t) \leq c'$. If $(\hat{z}, t) \in \mathcal{S} \triangleq \{(\hat{z}, t) \in \mathcal{Z}\times\mathbb{R}_+ | \xi(t) < \varepsilon', b(t) \leq c', t \geq t_{\text{avg}}, \sigma(e,t) = 2\}$ and the velocity measurement is available, the observer switches to the EKF, where $t_{\text{avg}} = \bar{\tau}_D(N_\sigma(t,0) - N_0)$. The convergence rate of $\xi(t)$ can be determined empirically for implementation purposes. After time T_2, defined in Theorem 1, only the second and third conditions are required for switching due to the practical stability of the switched system. The asymptotically stable observer and the norm estimator run at all time steps as a safeguard similar to [10].

V. Simulation Results

A simulation study is performed to test the performance of the switched observer compared to the individual observers in Sections IV-A and IV-B. Since, the EKF observer requires velocity measurements, a ZOH is assumed for velocities when they are unavailable. The initial state for simulation is selected as $z = [0.2, 0.3, 0.2]^\top$ and the velocities are $v = \left[0.4\mathrm{c}(\frac{\pi t}{4}), 0.5\mathrm{s}(\frac{\pi t}{4}), -0.4\mathrm{c}(\frac{\pi t}{4}) + 0.3\mathrm{s}(\frac{\pi t}{2})\right]^\top$, $\omega = \left[0, 0.1\mathrm{s}(\frac{\pi t}{8}), 0.1\mathrm{c}(\frac{\pi t}{4})\right]^\top$, where $\mathrm{c} = \cos$ and $\mathrm{s} = \sin$. White Gaussian noise with a mean of 0.001 and standard deviation 0.005 is added to the velocities and zero mean white Gaussian noise with standard deviation 0.01 is added to the state measurements. Ten randomly sampled times from $[0, 25]$s are chosen for the availability of velocity measurements. The initial estimate is selected as $\hat{z}(t_0) = [0.2, 0.3, 1.5]^\top$. The constants $\bar{\tau} = 5.4$ with $\ln(\mu) = 6.53$, $(\underline{\lambda}/2\bar{c}) - \epsilon = 1.2$, $k_b = 0.01$, the EKF gains $\bar{\alpha} = 1.3$, $R = 0.003\mathrm{I}_2$, $W = 2\mathrm{I}_3$ and the asymptotic observer gains $\Gamma = 2.6\,\mathrm{I}_2$, $k_2 = 0.93$ are tuned empirically to obtain the best performance. The estimation is started with the asymptotically stable observer and only switches to the EKF when the velocity measurements are available, $b(t) \leq 1.5$, and the average dwell time condition is satisfied. The above condition for $b(t)$ is empirically chosen since determining the region of convergence of the EKF is non-trivial. Fig. 1(a) shows faster convergence of the switched observer, which converges at $t = 9.1$s in comparison to the asymptotically stable observer, which converges at $t = 26.3$s. It is also observed that the EKF with ZOH for velocity measurements becomes unstable at the switching instant when the velocities are available. In Fig. 1(a), the EKF with ZOH error norm (solid green line) coincides with the switching dashed line as it becomes unstable. The dashed gray lines show the switching instants with the EKF active between $t = [5.1, 8.3)$s and $t = [17, 30]$s for the switched observer. Fig. 1(b) shows the evolution of the state estimate

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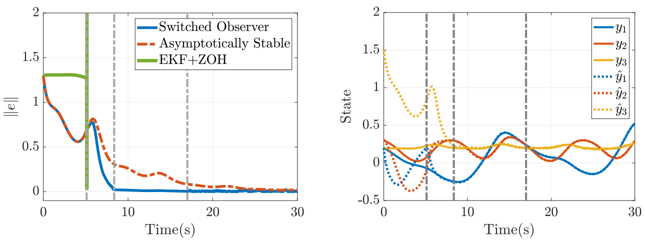

(a) (b)

Figure 1. Simulation Results showing (a) Comparison of observers based on error norm. (b) State estimation using switched observer.

compared to the ground truth using the switched observer. The steady state RMSE for the switched observer is 0.0028 and that of the asymptotically stable observer is 0.0074 demonstrating the robustness of the switched observer to measurement noise.

VI. Conclusion

The problem of switching between the UUB system and the asymptotically stable system with asymptotically decaying perturbation is considered in this paper. The multiple Lyapunov function-based stability analysis yields UUB stability of the switched system when an average dwell time condition is satisfied. The developed stability results are used for state estimation of the PDS using switched observers based on the availability of velocity measurements. Switching between the observers is shown to be UUB if the observability and average dwell time conditions are satisfied.

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